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Sylvester's double sums: an inductive proof of the general case

Teresa Krick

Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires and IMAS, CONICET, 1428 Buenos Aires, Argentina.

Agnes Szanto

Department of Mathematics, North Carolina State University, Raleigh NC 27695, USA.

Abstract

In 1853 J. Sylvester introduced a family of double sum expressions for two finite sets of indeterminates and showed that some members of the family are essentially the polynomial subresultants of the monic polynomials associated with these sets. In 2009, in a joint work with C. D'Andrea and H. Hong we gave the complete description of all the members of the family as expressions in the coefficients of these polynomials. In 2010, M.-F. Roy and A. Szpirglas presented a new and natural inductive proof for the cases considered by Sylvester. Here we show how induction also allows to obtain the full description of Sylvester's double-sums.

Key words: Sylvester's double sums, Subresultants.

1. Introduction

Let A and B be non-empty finite lists (ordered sets) of distinct indeterminates. In [Sylvester(1853)], J. Sylvester introduced for each $0 \le p \le |A|$ and $0 \le q \le |B|$ the following univariate polynomial in the variable x, of degree $\le p+q$, called the *double-sum expression* in A and B:

$$\operatorname{Sylv}^{p,q}(A,B) := \sum_{\substack{A' \subset A,\, B' \subset B \\ |A'|=p,\, |B'|=q}} R(x,A')\, R(x,B')\, \frac{R(A',B')\, R(A-A',B-B')}{R(A',A-A')\, R(B',B-B')},$$

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Email addresses: `krick@dm.uba.ar` (Teresa Krick), `aszanto@ncsu.edu` (Agnes Szanto).

URLs: `http://mate.dm.uba.ar/~ krick` (Teresa Krick), `http://www4.ncsu.edu/~ aszanto` (Agnes Szanto).

where for sets Y, Z of indeterminates,

$$R(Y,Z) := \prod_{y\in Y, z\in Z}(y-z), \qquad R(y,Z) := \prod_{z\in Z}(y-Z).$$

and by convention $R(Y,\emptyset)=1$.

Let now f, g be monic univariate polynomials such that

$$f = \prod_{\alpha\in A}(x-\alpha) = x^m + a_{m-1}x^{m-1} + \ldots + a_0 \quad\text{and}\quad g = \prod_{\beta\in B}(x-\beta) = x^n + b_{n-1}x^{n-1} + \ldots + b_0,$$

where $m := |A| \geq 1$ and $n := |B| \geq 1$. The k*-th subresultant* of the polynomials f and g is defined, for $0 \leq k < \min\{m,n\}$ or $k = \min\{m,n\}$ when $m \neq n$, as

$$\mathrm{Sres}_k(f,g) := \det \begin{array}{|cccc|l}
\multicolumn{4}{c}{\scriptstyle m+n-2k} & \\
\cline{1-4}
a_m & \cdots & \cdots\ a_{k+1-(n-k-1)} & x^{n-k-1}f(x) & \\
 & \ddots & \vdots & \vdots & \scriptstyle n-k \\
 & a_m\ \cdots & a_{k+1} & x^0 f(x) & \\
\cline{1-4}
b_n & \cdots & \cdots\ b_{k+1-(m-k-1)} & x^{m-k-1}g(x) & \\
 & \ddots & \vdots & \vdots & \scriptstyle m-k \\
 & b_n\ \cdots & b_{k+1} & x^0 g(x) & \\
\cline{1-4}
\end{array} \tag{1}$$

with $a_\ell = b_\ell = 0$ for $\ell < 0$. For $k=0$, $\mathrm{Sres}_0(f,g)$ coincides with the resultant:

$$\mathrm{Res}(f,g) = \prod_{\alpha\in A} g(\alpha) = (-1)^{mn}\prod_{\beta\in B} f(\beta). \tag{2}$$

Also, for instance,

$$\mathrm{Sres}_m(f,g) = f \text{ for } m<n \quad\text{and}\quad \mathrm{Sres}_n(f,g) = g \text{ for } n<m. \tag{3}$$

Relating Sylsvester's double sums with the polynomials f and g, it is immediate that

$$\mathrm{Sylv}^{0,0}(A,B) = R(A,B) = \mathrm{Res}(f,g), \tag{4}$$

$$\mathrm{Sylv}^{m,0}(A,B) = R(x,A) = f \quad\text{and}\quad \mathrm{Sylv}^{0,n}(A,B) = R(x,B) = g, \tag{5}$$

$$\mathrm{Sylv}^{m,n}(A,B) = R(x,A)\,R(x,B)\,R(A,B) = \mathrm{Res}(f,g)\,f\,g. \tag{6}$$

More generally, every value of the polynomial $\mathrm{Sylv}^{p,q}(A,B)$, which is symmetric in the α's and in the β's, can be expressed as a polynomial in x whose coefficients are rational functions in the a_i's and the b_j's. Sylvester in [Sylvester(1853)] gave this rational expression for the following values of (p,q):

(1) If $0 \leq k := p+q < \min\{m,n\}$ or if $k = m < n$, then [Sylvester(1853), Art. 21]:

$$\mathrm{Sylv}^{p,q}(A,B) = (-1)^{p(m-k)}\binom{k}{p}\mathrm{Sres}_k(f,g).$$

(2) If $p+q=m=n$, then [Sylvester(1853), Art. 22]:

$$\mathrm{Sylv}^{p,q}(A,B)=\binom{m-1}{q}f+\binom{m-1}{p}g.$$

(3) If $m<p+q<n-1$, then [Sylvester(1853), Arts. 23 & 24]):

$$\mathrm{Sylv}^{p,q}(A,B)=0.$$

(4) If $m<p+q=n-1$, then [Sylvester(1853), Art. 25]: $\mathrm{Sylv}^{p,q}(A,B)$ is a "numerical multiplier" of f, but the ratio is not established.

In [Lascoux and Pragacz(2003), Th.0.1 and Prop. 2.9], A. Lascoux and P. Pragacz presented new proofs for the cases covered by Items (1) and (2). More recently, in a joint work with C. D'Andrea and H. Hong in [D'Andrea et al.(2009), Th.2.10] we introduced a unified matrix formulation that allowed us to give an explicit formula for all possible values of (p,q), i.e. for $0\le p\le m, 0\le q\le n$. The proofs there were elementary though cumbersome. In 2010, M.-F. Roy and A. Szpirglas, were able to produce in [Roy and Szpirglas(2010), Main theorem] a new and natural inductive proof also for the cases covered by Item (1) and (2). The aim of this note is to give, inspired by [Roy and Szpirglas(2010)], an elementary inductive proof for all the cases. We furthermore show how the cases $p+q>\min\{m,n\}$, which seem somehow less natural since there is no "natural" expression associated to them (and were therefore not previously considered by Lascoux and Pragasz and Roy and Szpirglas) immediately yield simple proofs for other known interesting cases, as for instance for the cases $p+q=m<n$ and $p+q=m=n$, which didn't have simple proofs yet.

Let us now introduce the necessary notation to formulate our main result.
As in [D'Andrea et al.(2009)], we split the last column of the matrix in (1) to write $\mathrm{Sres}_k(f,g)$ as the sum of two determinants, obtaining an expression

$$\mathrm{Sres}_k(f,g)=F_k(f,g)\,f+G_k(f,g)\,g \tag{7}$$

where the polynomials $F_k(f,g)$ and $G_k(f,g)$ are defined for $0\le k<\min\{m,n\}$ or $k=\min\{m,n\}$ when $m\ne n$ as the determinants of the $(m+n-2k)$-matrices:

$$F_k(f,g):=\det\left|\begin{array}{ccccc} a_m & \cdots & \cdots & a_{k+1-(n-k-1)} & x^{n-k-1}\\ & \ddots & & \vdots & \vdots\\ & & a_m\ \cdots & a_{k+1} & x^0\\ \hline b_n & \cdots & \cdots & b_{k+1-(m-k-1)} & 0\\ & \ddots & & \vdots & \vdots\\ & & b_n\ \cdots & b_{k+1} & 0 \end{array}\right|\begin{array}{l} \\ n-k\\ \\ \\ m-k\\ \\ \end{array},\quad G_k(f.g):=\det\left|\begin{array}{ccccc} a_m & \cdots & \cdots & a_{k+1-(n-k-1)} & 0\\ & \ddots & & \vdots & \vdots\\ & & a_m\ \cdots & a_{k+1} & 0\\ \hline b_n & \cdots & \cdots & b_{k+1-(m-k-1)} & x^{m-k-1}\\ & \ddots & & \vdots & \vdots\\ & & b_n\ \cdots & b_{k+1} & x^0 \end{array}\right|\begin{array}{l} \\ n-k\\ \\ \\ m-k\\ \\ \end{array}.$$

We observe that when $k<\min\{m,n\}$, $\deg F_k(f,g)\le n-k-1$ and $\deg G_k(f,g)\le m-k-1$. Also

$$F_m(f,g)=1,\quad G_m(f,g)=0\ \text{ for } m<n\ \text{ and }\ F_n(f,g)=0,\ G_n(f,g)=1\ \text{ for } n<m \tag{8}$$

$$G_{m-1}(f,g)=1 \text{ for } m\le n\ \text{ and }\ F_{n-1}(f,g)=(-1)^{m-n+1} \text{ for } n\le m. \tag{9}$$

We finally introduce the following notation that we will keep all along in this text. Given $m,n\in\mathbb{N}$, $p,q\in\mathbb{Z}$ such that $0\le p\le m$, $0\le q\le n$ and $k=p+q$, we set

$$\overline{p}:=m-p,\quad \overline{q}:=n-q\ \text{ and }\ \overline{k}:=\overline{p}+\overline{q}-1=m+n-k-1.$$

Sylvester's double sums, for k "too big" w.r.t. m and n, will be expressed in our result in terms of the polynomials $F_{\overline{k}}(f,g)$ and $G_{\overline{k}}(f,g)$, well-defined since the condition $n-1\le k\le m+n-1$ for

$m < n$ is equivalent to $0 \le \overline{k} \le m$, and the condition $m \le k \le 2m-1$ for $m = n$ is equivalent to $0 \le \overline{k} \le m-1$.

Theorem 1. (See also [D'Andrea et al.(2009), Th.2.10])
Set $1 \le m \le n$, and let $0 \le p \le m$, $0 \le q \le n$ and $k = p+q$.
Then, for $(p,q) \ne (m,n)$,
– when $m < n$:

$$\mathrm{Sylv}^{p,q}(A,B) = \begin{cases} (-1)^{p(m-k)}\binom{k}{p}\mathrm{Sres}_k(f,g) & \text{for } 0 \le k \le m \\ 0 & \text{for } m+1 \le k \le n-2 \text{ when } m \le n-3 \\ (-1)^c\Big(\binom{\overline{k}}{\overline{p}}F_{\overline{k}}(f,g)\,f - \binom{\overline{k}}{\overline{q}}G_{\overline{k}}(f,g)\,g\Big) & \text{for } n-1 \le k \le m+n-1 \end{cases}$$

– when $m = n$:

$$\mathrm{Sylv}^{p,q}(A,B) = \begin{cases} (-1)^{p(m-k)}\binom{k}{p}\mathrm{Sres}_k(f,g) & \text{for } 0 \le k \le m-1 \\ (-1)^c\Big(\binom{\overline{k}}{\overline{p}}F_{\overline{k}}(f,g)\,f - \binom{\overline{k}}{\overline{q}}G_{\overline{k}}(f,g)\,g\Big) & \text{for } m \le k \le 2m-1 \end{cases},$$

where $c := \overline{p}\,\overline{q} + n - p - 1 + nq$;
and for $(p,q) = (m,n)$:

$$\mathrm{Sylv}^{m,n}(A,B) = \mathrm{Res}(f,g)\,f\,g.$$

Theorem 1 can be written in a more uniform manner instead of being split in cases: by Identity (7), for $0 \le k \le m$ when $m < n$ and for $0 \le k < m$ when $m = n$,

$$\mathrm{Sylv}^{p,q}(A,B) = (-1)^{p(m-k)}\Big(\binom{k}{p}F_k(f,g)f + \binom{k}{q}G_k(f,g)g\Big),$$

or for $0 \le \overline{k} \le m$ when $m < n$ and for $0 \le \overline{k} < m$, when $m = n$,

$$\begin{aligned}\mathrm{Sylv}^{p,q}(A,B) &= (-1)^c\Big(\binom{\overline{k}}{\overline{p}}\mathrm{Sres}_{\overline{k}}(f,g) - \binom{\overline{k}+1}{\overline{q}}G_{\overline{k}}(f,g)\,g\Big) \\ &= (-1)^c\Big(\binom{\overline{k}+1}{\overline{p}}F_{\overline{k}}(f,g)\,f - \binom{\overline{k}}{\overline{q}}\mathrm{Sres}_{\overline{k}}(f,g)\Big). \qquad (10)\end{aligned}$$

The cases "in between", for $m+1 \le k \le n-2$ when $m \le n-3$, are the cases when neither $0 \le k \le m$ nor $0 \le \overline{k} \le m$, i.e. the cases when the corresponding matrices F_k, G_k and $F_{\overline{k}}$, $G_{\overline{k}}$ are not defined (or could be defined as 0 for uniformity).

We also note that the case $k = m = n-1$ is covered twice: $\mathrm{Sres}_m(f,g) = f = F_m(f,g)f - G_m(f,g)g$ since $\overline{k} = m$, $F_m = 1$ and $G_m = 0$. Finally the case $p = m$, $q = n$ is Identity (6).

The proof of Theorem 1 is based, as the proof in [Roy and Szpirglas(2010)], on specialization properties.

2. Specialization properties

The following specialization property of Sylvester's double sums is well-known and proved in [Lascoux and Pragacz(2003), Lemma 2.8]. It is also reproved in [Roy and Szpirglas(2010), Prop.3.1], where it is used as one of the key ingredients of their inductive proof for the cases $k \le m < n$ and $k < m = n$. We repeat it here for sake of completeness.

Lemma 2. For any $\alpha \in A$ and $\beta \in B$,

- $\mathrm{Sylv}^{p,q}(A,B)(\alpha) = (-1)^p \, \mathrm{coeff}_{p+q}\big(\mathrm{Sylv}^{p,q}(A-\alpha,B)\big)\, R(\alpha,B)$ for $0 \le p < m$ and $0 \le q \le n$,
- $\mathrm{Sylv}^{p,q}(A,B)(\beta) = (-1)^{q+\overline{p}} \, \mathrm{coeff}_{p+q}\big(\mathrm{Sylv}^{p,q}(A,B-\beta)\big)\, R(\beta,A)$ for $0 \le p \le m$ and $0 \le q < n$.

Here coeff_{p+q} denotes the coefficient of order $p+q$ of $\mathrm{Sylv}^{p,q}(A,B-\beta)$.

Proof.

$$\begin{aligned}
\mathrm{Sylv}^{p,q}(A,B)(\alpha) &= \sum_{\substack{A' \subset A-\alpha,\ B' \subset B \\ |A'|=p,\ |B'|=q}} R(\alpha,A')\, R(\alpha,B')\, \frac{R(A',B')\, R(A-A',B-B')}{R(A',A-A')\, R(B',B-B')} \\
&= (-1)^p\, R(\alpha,B) \sum_{\substack{A' \subset A-\alpha,\ B' \subset B \\ |A'|=p,\ |B'|=q}} \frac{R(A',B')\, R((A-\alpha)-A',B-B')}{R(A',(A-\alpha)-A')\, R(B',B-B')} \\
&= (-1)^p\, \mathrm{coeff}_{p+q}\big(\mathrm{Sylv}^{p,q}(A-\alpha,B)\big)\, R(\alpha,B).
\end{aligned}$$

The second identity is a consequence of the fact that

$$\mathrm{Sylv}^{p,q}(A,B) = (-1)^{pq}\, (-1)^{\overline{p}\,\overline{q}}\, \mathrm{Sylv}^{q,p}(B,A).$$

□

In the following we replace the specialization property of subresultants proved in [Roy and Szpirglas(2010), Prop. 4.1] by the specialization property of the polynomials F_k and G_k. This will allow a more uniform and simpler proof of our main theorem, covering all cases of p and q.

Lemma 3. For any root α of f and any root β of g, we have

- $F_k(f,g)(\beta) = -\,\mathrm{coeff}_{n-k-1}\big(F_{k-1}(f, \frac{g}{x-\beta})\big)$ for $1 \le k \le \min\{m,n\}-1$,
- $G_k(f,g)(\alpha) = (-1)^{m-k-1}\, \mathrm{coeff}_{m-k-1}\big(G_{k-1}(\frac{f}{x-\alpha}, g)\big)$ for $1 \le k \le \min\{m,n\}-1$.

Here coeff_{n-k-1} (resp. coeff_{m-k-1}) denotes the coefficient of order $n-k-1$ (resp. $m-k-1$) of the corresponding polynomial.

Proof. Given a root β of g, we set

$$\frac{g}{x-\beta} := x^{n-1} + b'_{n-2}x^{n-2} + \cdots + b'_0.$$

The following relationship between the coefficients of g and of $\frac{g}{x-\beta}$ is straightforward:

$$b_i = b'_{i-1} - \beta b'_i \text{ for } 1 \le i \le n-1 \text{ and } b_0 = -\beta b'_0. \tag{11}$$

(Here $b_n = b'_{n-1} = 1$.)

First consider

$$\mathrm{coeff}_{n-k-1}\Big(F_{k-1}(f,\frac{g}{x-\beta})\Big) = \mathrm{coeff}_{n-k-1}\Big(\det \begin{array}{|ccccc|l} \cline{1-5} a_m & \cdots & \cdots & a_{k-(n-k-1)} & x^{n-k-1} & \\ & \ddots & & \vdots & \vdots & {\scriptstyle (n-1)-(k-1)} \\ & & a_m \ \cdots & a_k & x^0 & \\ \cline{1-5} b'_{n-1} & \cdots & \cdots & b'_{k-(m-k)} & 0 & \\ & \ddots & & \vdots & \vdots & {\scriptstyle m-(k-1)} \\ & & b'_{n-1} \ \cdots & b'_k & 0 & \\ \cline{1-5} \end{array} \Big)$$

$$= (-1)^{m+n} \det \begin{array}{|ccccc|l} \cline{1-5} 0 & a_m & \cdots & \cdots & a_{k-(n-k-2)} & \\ & & \ddots & & \vdots & {\scriptstyle n-1-k} \\ & & & a_m \ \cdots & a_k & \\ \cline{1-5} b'_{n-1} & \cdots & \cdots & \cdots & b'_{k-(m-k)} & \\ & \ddots & & & \vdots & {\scriptstyle m-k+1} \\ & & b'_{n-1} \ \cdots & & b'_k & \\ \cline{1-5} \end{array} = (-1)^{m-k+1} \det \begin{array}{|cccc|l} \cline{1-4} a_m & \cdots & \cdots & a_{k-(n-k-2)} & \\ & \ddots & \vdots & & {\scriptstyle n-1-k} \\ & a_m \ \cdots & & a_k & \\ \cline{1-4} b'_{n-1} & \cdots & \cdots & b'_{k-(m-k-1)} & \\ & \ddots & \vdots & & {\scriptstyle m-k} \\ & b'_{n-1} \ \cdots & & b'_k & \\ \cline{1-4} \end{array} .$$

We apply elementary column operations on the matrix above, replacing the j-th column C_j by $C_j - \beta C_{j-1}$ starting from the last column $C_{n+m-2k-1}$ up to the second column C_2, and using the relations in (11):

$$\mathrm{coeff}_{n-k-1}\Big(F_{k-1}(f,\frac{g}{x-\beta})\Big) = (-1)^{m-k+1} \det \begin{array}{|cccccc|l} \cline{1-6} a_m & a_{m-1}-\beta a_m & \cdots & & \cdots & a_{k-(n-k-2)} - \beta a_{k+1-(n-k-2)} & \\ & \ddots & & & \vdots & & {\scriptstyle n-1-k} \\ & & a_m & a_{m-1}-\beta a_m & \cdots & a_k - \beta a_{k+1} & \\ \cline{1-6} b_n & b_{n-1} & \cdots & & \cdots & b_{k+1-(m-k-1)} & \\ & \ddots & & & \vdots & & {\scriptstyle m-k} \\ & & b_n & b_{n-1} & \cdots & b_{k+1} & \\ \cline{1-6} \end{array} . \tag{12}$$

Next consider

$$F_k(f,g)(\beta) = \det \begin{array}{|ccccc|l} \cline{1-5} a_m & \cdots & \cdots & a_{k+1-(n-k-1)} & \beta^{n-k-1} & \\ & \ddots & & \vdots & \vdots & {\scriptstyle n-k} \\ & & a_m \ \cdots & a_{k+1} & \beta^0 & \\ \cline{1-5} b_n & \cdots & \cdots & b_{k+1-(m-k-1)} & 0 & \\ & \ddots & & \vdots & \vdots & {\scriptstyle m-k} \\ & & b_n \ \cdots & b_{k+1}k & 0 & \\ \cline{1-5} \end{array} .$$

We apply elementary row operations on the matrix above, replacing the i-th row R_i by $R_i - \beta R_{i+1}$,

starting from the first row R_1 up to row R_{n-k-1}:

$$F_k(f,g)(\beta) = \det \left|\begin{array}{cccccc|l} a_m & a_{m-1}-\beta a_m & & & \cdots & a_{k+1-(n-k-1)}-\beta a_{k+2-(n-k-1)} & 0 \\ & \ddots & & & & \vdots & \vdots \\ & & a_m & a_{m-1}-\beta a_m & \cdots & a_{k+2}-\beta a_{k+1} & 0 \\ & & & a_m & a_{m-1}\ \cdots & a_{k+1} & 1 \\ \hline b_n & \cdots & & & \cdots & b_{k+1-(m-k-1)} & 0 \\ & \ddots & & & & \vdots & \vdots \\ & & & b_n & \cdots & b_{k+1} & 0 \end{array}\right| \begin{array}{l} \\ {\scriptstyle n-k} \\ \\ \\ \\ {\scriptstyle m-k} \\ \\ \end{array}$$

$$= (-1)^{m-k} \det \left|\begin{array}{ccccc} a_m & a_{m-1}-\beta a_m & & \cdots & a_{k+1-(n-k-1)}-\beta a_{k+2-(n-k-1)} \\ & \ddots & & & \vdots \\ & & a_m \ \ a_{m-1}-\beta a_m & \cdots & a_{k+2}-\beta a_{k+1} \\ \hline b_n & \cdots & & \cdots & b_{k+1-(m-k-1)} \\ & \ddots & & & \vdots \\ & & b_n & \cdots & b_{k+1} \end{array}\right| \begin{array}{l} \\ {\scriptstyle n-k-1} \\ \\ \\ {\scriptstyle m-k} \\ \\ \end{array} . \quad (13)$$

We obtain the first identity of the statement by comparing (12) and (13).
For the second identity, we have

$$\begin{aligned} G_k(f,g)(\alpha) &= (-1)^{(n-k)(m-k)} F_k(g,f)(\alpha) \\ &= (-1)^{(n-k)(m-k)+1} \mathrm{coeff}_{m-k-1}\big(F_{k-1}(g, \frac{f}{x-\alpha})\big) \\ &= (-1)^{(n-k)(m-k)+1} (-1)^{(m-k)(n-k+1)} \mathrm{coeff}_{m-k-1}\big(G_{k-1}(\frac{f}{x-\alpha}, g)\big) \\ &= (-1)^{m-k-1} \mathrm{coeff}_{m-k-1}\big(G_{k-1}(\frac{f}{x-\alpha}, g)\big). \end{aligned}$$

□

As an immediate consequence we obtain the specialization property of subresultants which seemed to have been stated and proved for the first time in [Roy and Szpirglas(2010), Prop. 4.1].

Corollary 4. For any root α of f, any root β of g and any $0 \le k < \min\{m,n\}$, we have

- $\mathrm{Sres}_k(f,g)(\beta) = (-1)^{m-k}\,\mathrm{coeff}_k\big(\mathrm{Sres}_k(f\,,\,\frac{g}{x-\beta})\big)\, f(\beta)$,
- $\mathrm{Sres}_k(f,g)(\alpha) = \mathrm{coeff}_k\big(\mathrm{Sres}_k(\frac{f}{x-\alpha}\,,\,g)\big)\, g(\alpha)$.

Here coeff_k denotes the coefficient of order k of the corresponding polynomial.

Proof. It is sufficient to prove the first identity, since the second identity is a consequence of

$$\mathrm{Sres}_k(g,f) = (-1)^{(m-k)(n-k)} \mathrm{Sres}_k(f,g).$$

By (7) and the previous lemma,

$$\mathrm{Sres}_k(f,g)(\beta) = F_k(f,g)(\beta)\, f(\beta) = -\,\mathrm{coeff}_{n-k-1}\big(F_{k-1}(f, \frac{g}{x-\beta})\big)\, f(\beta).$$

Now it is immediate to verify by the definition of the principal scalar subresultant of order k that

$$\mathrm{coeff}_{n-k-1}\big(F_{k-1}(f,\frac{g}{x-\beta})\big)=(-1)^{m-k-1}\mathrm{coeff}_k\big(\mathrm{Sres}_k(f,\frac{g}{x-\beta})\big).$$

□

3. Proof of Theorem 1

It turns out that the cases of Theorem 1 where k is "big" are easy to prove by induction and will be used later in the other cases. That is why we start with this case first in the following proposition. The proof will use a lemma for the extremal cases (p,n) and (m,q), which is given after the proposition. We recall that $\overline{p}=m-p$, $\overline{q}=n-q$, and $\overline{k}=m+n-k-1$.

Proposition 5. Set $1\le m\le n$ and let $0\le p\le m,\ 0\le q\le n$ and $k=p+q$ be such that $n-1\le k\le m+n-1$, i.e. $0\le\overline{k}\le m$, when $m<n$ or $m\le k\le 2m-1$, i.e. $0\le\overline{k}\le m-1$, when $m=n$. Then

$$\mathrm{Sylv}^{p,q}(A,B)=(-1)^{\overline{p}\,\overline{q}+n-p-1+nq}\Big(\binom{\overline{k}}{\overline{p}}F_{\overline{k}}(f,g)\,f-\binom{\overline{k}}{\overline{q}}G_{\overline{k}}(f,g)\,g\Big).$$

Proof. By induction on $\overline{k}\ge 0$:
The case $\overline{k}=0$ implies $(p,q)=(m-1,n)$ or $(p,q)=(m,n-1)$ and will follow from Lemma 6.

Now set $\overline{k}>0$.
– For $p=m$ and $q<n$ or $p<m$ and $q=n$, also by Lemma 6,

$$\mathrm{Sylv}^{m,q}(A,B)=(-1)^{n-m-1+nq}F_{\overline{q}-1}(f,g)f\quad\text{and}\quad \mathrm{Sylv}^{p,n}(A,B)=(-1)^{p}G_{\overline{p}-1}(f,g)g$$

accordingly, which matches the statement since in these cases $\binom{\overline{k}}{\overline{q}}$ or $\binom{\overline{k}}{\overline{p}}$ equals 0.
– For $p<m$ and $q<n$, we specialize $\mathrm{Sylv}^{p,q}(A,B)$ of degree $k\le m+n-2$ in the $m+n$ elements of $A\cup B$ by means of Lemma 2 and the inductive hypothesis:

$$\begin{aligned}\mathrm{Sylv}^{p,q}(A,B)(\alpha)&=(-1)^{p}\,\mathrm{coeff}_k\big(\mathrm{Sylv}^{p,q}(A-\alpha,B)\big)\,g(\alpha)\\&=(-1)^{c'+p}\,\mathrm{coeff}_k\Big(\binom{\overline{k}-1}{\overline{p}-1}\mathrm{Sres}_{\overline{k}-1}(\frac{f}{x-\alpha},g)-\binom{\overline{k}}{\overline{q}}G_{\overline{k}-1}(\frac{f}{x-\alpha},g)g\Big)\,g(\alpha),\end{aligned}$$

by Identity (10). Here $c'=(\overline{p}-1)\overline{q}+n-p-1+nq$.
Note that we are looking for the coefficient of degree k of the expression between brackets; the condition $\overline{k}-1\le m-1<n-1\le k$ in case $m<n$ and $\overline{k}-1\le m-2<k$ in case $m=n$ imply in both cases that $\deg(\mathrm{Sres}_{\overline{k}-1}(\frac{f}{x-\alpha},g))\le\overline{k}-1<k$. Then

$$\mathrm{Sylv}^{p,q}(A,B)(\alpha)=(-1)^{c'+p}\,\mathrm{coeff}_k\Big(-\binom{\overline{k}}{\overline{q}}G_{\overline{k}-1}(\frac{f}{x-\alpha},g)g\Big)\,g(\alpha).$$

When $\overline{k}-1<m-1$, i.e $k\ge n$, we apply Lemma 3 and get

$$\begin{aligned}\mathrm{Sylv}^{p,q}(A,B)(\alpha)&=(-1)^{c'+p}\Big(-\binom{\overline{k}}{\overline{q}}\,\mathrm{coeff}_{k-n}\big(G_{\overline{k}-1}(\frac{f}{x-\alpha},g)\big)\,g(\alpha)\Big)\\&=(-1)^{c'+p+k-n}\Big(-\binom{\overline{k}}{\overline{q}}G_{\overline{k}}(f,g)(\alpha)g(\alpha)\Big)\\&=(-1)^{\overline{p}\,\overline{q}+n-p-1+nq}\Big(-\binom{\overline{k}}{\overline{q}}G_{\overline{k}}(f,g)(\alpha)g(\alpha)\Big).\end{aligned}$$

When $\overline{k}-1=m-1$, $G_{\overline{k}-1}(\frac{f}{x-\alpha},g)=0=G_{\overline{k}}(f,g)$ and therefore we also get

$$\mathrm{Sylv}^{p,q}(A,B)(\alpha)=(-1)^{\overline{p}\,\overline{q}+n-p-1+nq}\Big(-\binom{\overline{k}}{\overline{q}}G_{\overline{k}}(f,g)(\alpha)g(\alpha)\Big).$$

Analogously,

$$\begin{aligned}\mathrm{Sylv}^{p,q}(A,B)(\beta)&=(-1)^{q+\overline{p}+c''}\,\mathrm{coeff}_k\Big(\binom{\overline{k}}{\overline{p}}F_{\overline{k}-1}(f,\frac{g}{x-\beta})f-\binom{\overline{k}-1}{\overline{q}-1}\mathrm{Sres}_{\overline{k}-1}(f,\frac{g}{x-\beta}))\Big)\,f(\beta)\\&=(-1)^{q+\overline{p}+c''}\,\mathrm{coeff}_k\Big(\binom{\overline{k}}{\overline{p}}F_{\overline{k}-1}(f,\frac{g}{x-\beta})f\Big)\,f(\beta)\\&=(-1)^{q+\overline{p}+c''}\binom{\overline{k}}{\overline{p}}\,\mathrm{coeff}_{k-m}\big(F_{\overline{k}-1}(f,\frac{g}{x-\beta})\big)\,f(\beta)\\&=(-1)^{q+\overline{p}+c''+1}\binom{\overline{k}}{\overline{p}}\,F_{\overline{k}}(f,g)(\beta)\,f(\beta),\end{aligned}$$

where $c''=\overline{p}\,(\overline{q}-1)+n-1-p-1+(n-1)q$. Therefore,

$$\mathrm{Sylv}^{p,q}(A,B)(\beta)=(-1)^{\overline{p}\,\overline{q}+n-p-1+nq}\binom{\overline{k}}{\overline{p}}\,F_{\overline{k}}(f,g)(\beta)\,f(\beta).$$

This concludes the proof. $\square$

The next lemma covers the cases (p,n) and (m,q) needed in the proof of the previous result. Observe that

$$\mathrm{Sylv}^{p,n}(A,B)=g\sum_{A'\subset A,|A'|=p}R(x,A')\,\frac{R(A',B)}{R(A',A-A')}\quad\text{for}\quad p\le m,$$
$$\mathrm{Sylv}^{m,q}(A,B)=f\sum_{B'\subset B,|B'|=q}R(x,B')\frac{R(A,B')}{R(B',B-B')}\quad\text{for}\quad q\le n.$$

Lemma 6. Set $1\le m\le n$. Then
1. $\mathrm{Sylv}^{p,n}(A,B)=(-1)^pG_{\overline{p}-1}(f,g)\,g$ for $0\le p\le m-1$, i.e. $1\le\overline{p}\le m$.
2. $\mathrm{Sylv}^{m,q}(A,B)=(-1)^{n-m-1+nq}F_{\overline{q}-1}(f,g)\,f$ for $n-m-1\le q\le n-1$, i.e. $1\le\overline{q}\le m+1$, when $m<n$ and for $0\le q\le m-1$, i.e. $1\le\overline{q}\le m$, when $m=n$.

Proof. (1) By induction on $m\ge 1$.
The case $m=1$ is clear from Identities 5 and 9, since in this case $p=0$ and $\overline{p}=1$.
Now set $m>1$ and let $0\le p\le m-1$. Both $\mathrm{Sylv}^{p,n}(A,B)$ and $G_{\overline{p}-1}(f,g)\,g$ are polynomials of degree bounded by $p+n<m+n$ and we compare them by specializing them into the $m+n$ elements $\alpha\in A$ and $\beta\in B$. Clearly both expressions vanish at every $\beta\in B$ and so we only need to compare them at $\alpha\in A$.
– For $p<m-1$, we apply Lemma 2, the inductive hypothesis and Lemma 3 (and the fact that g is monic):

$$\begin{aligned}\mathrm{Sylv}^{p,n}(A,B)(\alpha)&=(-1)^p\,\mathrm{coeff}_{p+n}\big(\mathrm{Sylv}^{p,n}(A-\alpha,B)\big)\,g(\alpha)\\&=(-1)^{2p}\,\mathrm{coeff}_{p+n}\big(G_{(m-1)-p-1}(\frac{f}{x-\alpha},g)\,g\big)\,g(\alpha)\\&=\mathrm{coeff}_p\big(G_{(m-1)-p-1}(\frac{f}{x-\alpha},g)\big)\,g(\alpha)\;=\;(-1)^p\,G_{\overline{p}-1}(f,g)(\alpha)\,g(\alpha).\end{aligned}$$

– For $p = m-1$:

$$\begin{aligned}\mathrm{Sylv}^{p,n}(A,B)(\alpha) &= R(\alpha, A-\alpha)\frac{R(A-\alpha,B)}{R(A-\alpha,\alpha)}\,g(\alpha)\\ &= (-1)^{m-1}\prod_{\alpha'\in A} g(\alpha') \;=\; (-1)^{m-1}\mathrm{Res}(f,g) \;=\; (-1)^{m-1}G_0(f,g)(\alpha)g(\alpha),\end{aligned}$$

by Identity (2) and the fact that $\mathrm{Res}(f,g) = F_0(f,g)f + G_0(f,g)g$ has degree 0 in x. Therefore $\mathrm{Sylv}^{p,n}(A,B) = (-1)^p G_{\overline{p}-1}(f,g)\,g$.

(2) By induction on $n \geq m$.
For $n = m$, by Item (1) we have that for $0 \leq q \leq m-1$,

$$\begin{aligned}\mathrm{Sylv}^{m,q}(A,B) &= (-1)^{mq}\,\mathrm{Sylv}^{q,m}(B,A) \;=\; (-1)^{mq+q}G_{\overline{q}-1}(g,f)\,f\\ &= (-1)^{mq+q}(-1)^{(m-(\overline{q}-1))(n-(\overline{q}-1))}F_{\overline{q}-1}(f,g)\,f \;=\; (-1)^{nq-1}F_{\overline{q}-1}(f,g)\,f.\end{aligned}$$

Now set $n \geq m+1$ and let $n-m-1 \leq q \leq n-1$. Both $\mathrm{Sylv}^{m,q}(A,B)$ and $F_{\overline{q}-1}(f,g)\,f$ are polynomials of degree bounded by $m+q < m+n$ and we compare them by specializing them in the $m+n$ elements $\alpha \in A$ and $\beta \in B$. Clearly both expressions vanish at every $\alpha \in A$ and so we only need to compare them at $\beta \in B$.
– For $q < n-1$, we apply Lemma 2, the inductive hypothesis and Lemma 3:

$$\begin{aligned}\mathrm{Sylv}^{m,q}(A,B)(\beta) &= (-1)^q\,\mathrm{coeff}_{m+q}\big(\mathrm{Sylv}^{m,q}(A,B-\beta)\big)\,f(\beta)\\ &= (-1)^{q+(n-1-m-1)+(n-1)q}\,\mathrm{coeff}_{m+q}\Big(F_{(n-1)-q-1}(f,\frac{g}{x-\beta})\,f\Big)\,f(\beta)\\ &= (-1)^{(n+m-2+nq)+1}\,F_{\overline{q}-1}(f,g)(\beta)\,f(\beta).\end{aligned}$$

– For $q = n-1$,

$$\begin{aligned}\mathrm{Sylv}^{m,n-1}(A,B)(\beta) &= f(\beta)\,R(\beta,B-\beta)\frac{R(A,B-\beta)}{R(B-\beta,\beta)}\\ &= (-1)^{n-1+m(n-1)}\prod_{\beta'\in B} f(\beta') \;=\; (-1)^{(mn+n-m-1)+mn}\mathrm{Res}(f,g)\\ &= \;(-1)^{n-m-1}F_0(f,g)(\beta)f(\beta).\end{aligned}$$

Therefore $\mathrm{Sylv}^{m,q}(A,B) = (-1)^{n-m-1+nq}F_{\overline{q}-1}(f,g)\,f$ as wanted.
□

As a particular case of Proposition 5, using Identities (8) and (9), we obtain Case (2) and a particular case of Case (4) of the introduction:

Corollary 7.

(1) Set $1 \leq m = n$ and let $0 \leq p,\ 0 \leq q$ be such that $p+q = m$. Then

$$\mathrm{Sylv}^{p,q}(A,B) = \binom{m-1}{q}\,f + \binom{m-1}{p}\,g.$$

(2) Set $1 \leq m = n-2$ and let $0 \leq p \leq m$, $0 \leq q$ be such that $p+q = n-1$. Then

$$\mathrm{Sylv}^{p,q}(A,B) = (-1)^{p+1}\binom{m}{p}\,f.$$

This allows us to simplify the rather long proofs for the cases when $p+q=m<n$, which appeared previously in [Lascoux and Pragacz(2003)], [D'Andrea et al.(2007)] and [Roy and Szpirglas(2010)].

Proposition 8. Set $1 \le m \le n-1$ and let $p \ge 0,\ q \ge 0$ be such that $1 \le p+q=m$. Then

$$\mathrm{Sylv}^{p,q}(A,B)=\binom{m}{p} f.$$

Proof. By induction on $n \ge m+1$, comparing the two expressions at the $n>m$ elements of B. For $n=m+1$, by Lemma 2 and Corollary 7(1),

$$\begin{aligned}\mathrm{Sylv}^{p,q}(A,B)(\beta) &= \mathrm{coeff}_m\big(\mathrm{Sylv}^{p,q}(A,B-\beta)\big)\, f(\beta)\\ &= \mathrm{coeff}_m\Big(\binom{m-1}{q} f+\binom{m-1}{p}\frac{g}{x-\beta}\Big)\, f(\beta)\\ &= \Big(\binom{m-1}{q}+\binom{m-1}{p}\Big) f(\beta) \;=\; \binom{m}{p} f(\beta).\end{aligned}$$

Now set $n>m+1$,

$$\mathrm{Sylv}^{p,q}(A,B)(\beta) \;=\; \mathrm{coeff}_m\big(\mathrm{Sylv}^{p,q}(A,B-\beta)\big)\, f(\beta) \;=\; \mathrm{coeff}_m\Big(\binom{m}{p} f\Big)\, f(\beta) \;=\; \binom{m}{p} f(\beta).$$

□

We finish the proof of Theorem 1 by splitting it into the two remaining cases to be proven. The first case is the inductive proof of [Roy and Szpirglas(2010)] that we repeat here for the sake of completeness.

Proposition 9. Set $1 \le m \le n$ and let $p \ge 0,\ q \ge 0$ and $k=p+q$ be such that $k \le m$ when $m<n$ and $k<m$ when $m=n$. Then

$$\mathrm{Sylv}^{p,q}(A,B)=(-1)^{p(m-k)}\binom{k}{p}\mathrm{Sres}_k(f,g).$$

Proof. By induction on $m \ge 1$:
The case $m=1$ is completely covered by Identities (4), (5), (3) and Proposition 8.
Now set $m>1$ and let $0 \le k=p+q \le m$ if $m<n$ and $0 \le k=p+q<m$ if $m=n$. We have
– For $0 \le k \le m-1$, we compare $\mathrm{Sylv}^{p,q}(A,B)$ and $\mathrm{Sres}_k(f,g)$, which are both of degree $k<m$, by specializing them into the m elements $\alpha \in A$ by means of Lemma 2, the inductive hypothesis and Corollary 4:

$$\begin{aligned}\mathrm{Sylv}^{p,q}(A,B)(\alpha) &= (-1)^p\,\mathrm{coeff}_k\big(\mathrm{Sylv}^{p,q}(A-\alpha,B)\big)\, g(\alpha)\\ &= (-1)^p(-1)^{p(m-1-k)}\binom{k}{p}\,\mathrm{coeff}_k\Big(\mathrm{Sres}_k\big(\frac{f}{x-\alpha},g\big)\Big)\, g(\alpha)\\ &= (-1)^{p(m-k)}\binom{k}{p}\,\mathrm{Sres}_k(f,g)(\alpha).\end{aligned}$$

– For $k=m<n$, it is Proposition 8. □

Proposition 10. Set $1 \le m \le n-3$ and let $0 \le p \le m,\ 0 \le q \le n$ be such that $m+1 \le p+q \le n-2$. Then

$$\mathrm{Sylv}^{p,q}(A,B)=0.$$

Proof. By induction on $n \geq m+3$, specializing the expression in the $n > m+1 = k$ elements of B by Lemma 2.
For $n = m+3$, by Corollary 7(2):

$$\mathrm{Sylv}^{p,q}(A,B)(\beta) = -\,f(\beta)\mathrm{coeff}_{m+1}\big(\,\mathrm{Sylv}^{p,q}(A,B-\beta)\big) = -\,f(\beta)\mathrm{coeff}_{m+1}\Big((-1)^{p+1}\binom{m}{p}\,f\Big) = 0,$$

since $\deg(f) = m < m+1$.
The case $n > m+3$ follows immediately. □

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